数理解析研究所講究録
267巻 1976年 102-108



# REDUCIBILITY OF FUCHSIAN SYSTEMS

By Kenjiro Okubo, Department of Mathematics
Tokyo Metropolitan University.

**Definition.** A system of linear ordinary differential equations with rational coefficient:

$$(1) \qquad x' = Ax$$

is reducible, if and only if there is a non-singular linear transformation:

$$x = T(t)y$$

such that the transformed system

$$y' = By = (T^{-1}AT - T^{-1}T')y$$

has a reducible coefficient $B(t)$. A rational transformation $B(T)$ is reducible if it has a proper non-trivial invariant subspace $V$ of $C_n$ independent of $t$:

$$B(t)V < V \qquad \text{for all } t.$$

Of course, there is a constant non-singular matrix $C$, such that $C^{-1}B(t)C$ has a off diagonal block with all the element zero:

$$(2) \qquad C^{-1}B(t)C = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$$

if $B(t)$ is reducible.

**Definition.** Let $S=[\lambda_1,\ldots,\lambda_n,\infty]$ be the set of poles of $A(t)$. Let $X(t)$ be some fixed fundamental set of solutions of the system (1). Let $\gamma$ be a closed circuit on $P^1-S$, and let $X(t)M(\gamma)$ be the result of analytic continuation of $X(t)$ along $\gamma$. We call the representation of $\pi_1(P^1-S)$ in $GL(n,C)$ defined by $\gamma \to M(\gamma)$, the monodromy representation of (1) with respect to $X(t)$.

**Definition.** A linear representation of a group $G$ is reducible if there is a proper non-trivial invariant subspace for all the elements.

<u>Theorem 1.</u> If the system (1) is reducible, then every monodromy representation is reducible.

<u>Corollary.</u> If a monodromy representation is irreducible for (1), then (1) is irreducible. ( = not reducible ).

<u>Proof of the theorem.</u> By a rational transformation, we get a new system of the form:

$$y_1' = B_{11}y_1$$
$$y_2' = B_{21}y_1+B_{22}y_2$$

We have a non-trivial set of solutions for which $y_1$=o identically, that is, we have a fundamental set of solutions of the form:

$$Y(t) = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix}.$$

If $Y_{11}$ be an r by r matrix, $Y_{21}$ be (n-r) by r, $Y_{22}$ be (n-r) by (n-r), and the zero block be r by (n-r). Then it is clear that the representation with respect to this set has the form:

$$\begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}$$

Any vector whose first r components are zero is transformed by this type of matrix from the right into a vector whose first r components are zero.

<u>Theorem 2.</u> If a monodromy representation of the system (1) has an (n-1)dimensional invariant subspace, and it is Fuchsian, then the system itself is reducible.

Proof. We may assume that the invariant subspace V is the set of vectors whose n-th component[redacted] is zero. Let $g_1$ ,....,$g_p$ be the generators of the representation. They have the form:

$$g_j = \left(\begin{array}{c|c} ** & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * & e^{-2\pi i c_j} \end{array}\right)$$

where $c_j$ $(j=1,2,\dots,n)$ are some constants determined up to an integral difference. Let $x_n(t)$ be the n-th column vector of the fundamental set $X(t)$ corresponding to the above representation. Then by simple observation, we see the vector solution $x_n(t)$ is transformed into $x_n(t)\exp(-2\pi i c_j)$ by the circuit around $_j$. Hence the column vector $y(t)=r(t)x_n(y)$ with a scalar multiplier;

$$r(t) = (t-a_1)^{c_1}\dots\dots(t-a_m)^{c_m}$$

is single-valued in the entire complex plane. On the other hand, we have assumed the system to be a Fuchsian system, that is, $y(t)$ is a non-trivial rational function, satisfying the system of differential equations:

$$dy/dt = r'x_n(t) + rx_n'(t) = [\ r'/r + A\ ]y$$

Let $R(t)$ be any non-singular n by n matrix with rational elements whose n-th column vector is $y(t)$. Then the n-th column of the matrix:

$$dR/dt - A(t)R - r'/r\cdot R$$

is identically zero. Multiply $R^{-1}$ from the left, and we see the matrix

$$R^{-1}[R' - AR] - (r'/r)I$$

has zero n-th column. Consequently, the matrix

$$B(t) = R^{-1}[R' - AR]$$

has the n-th column of the form:

$$b_n(t) = (0,0,\dots,0,r'/r)$$

That is, $B(t)$ has an invariant subspace of vectors whose n-th component zero.

<u>Theorem 3</u>. Let

$$A = \begin{pmatrix} -a_1 & & & & 1 \\ & \ddots & & 0 & \vdots \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & -a_{n-1} & 1 \\ b_1 & \dots\dots & & b_{n-1} & -a_n \end{pmatrix} \qquad B = \mathrm{diag}[\ 0,0,\dots,0,1]$$

be two constant n by n matrices. We denote the eigenvalues of the matrix A by $c_1, c_2, \dots, c_n$. We consider the system of n first order linear differential equations:

$$(*) \quad (t-B)dx/dt = Ax$$

under the conditions:

$1^\circ.\ a_j \not\equiv 0 \pmod{1} \qquad 2^\circ.\ c_j \not\equiv 0 \pmod{1} \qquad 3^\circ.\ a_j - a_k \not\equiv 0 \pmod{1}$

for all j and k.

The system (*) has n singular solutions of the form:

$$x_j(t) = t^{-a_j} \sum_{m=0}^{\infty} g_j(m) t^m \qquad (\ j=1,2,\dots,n-1\ )$$

$$x_n(t) = (t-1)^{-a_n} \sum_{m=0}^{\infty} g_n(m)(t-1)^m .$$

These solutions constitue a fundamental set X(t) of solutions, with respect to which the monodromy has generators of the form:

$$M_o = \begin{pmatrix} e_1 & & & 0 & q_1(e_1-1) \\ & e_2 & & & \\ & & \ddots & & \\ & 0 & & e_{n-1} & q_{n-1}(e_{n-1}-1) \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \qquad (\ e_j = \exp(2\pi i(-a_j)).\ )$$

$$M_1 = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & 0 & & 1 & \\ p_1(e_n-1) & p_2(e_n-1) & \cdots & p_{n-1}(e_n-1) & e_n \end{pmatrix}$$

where 2(n-1) constants $p_j, q_k$ are given by the formulae:

$$p_j = -b_j \frac{\Gamma(1-a_j)\cdot\Gamma(a_n)\cdot\prod_{k\neq j,n}\Gamma(1-a_j+a_k)}{\prod_k \Gamma(1-a_j+c_k)}$$

$$q_j = -\frac{1}{b_j} \frac{\Gamma(a_j)\cdot\Gamma(1-a_n)\cdot\prod_{k\neq j,n}\Gamma(a_j-a_k)}{\prod_k \Gamma(c_k-a_j)}$$

Theorem 4. Under the conditions of the preceding theorem, the system (*) is irreducible.

Proof. We remark that none of the quantities $p_1,\dots p_{n-1},q_1,\dots,q_{n-1}$ is zero. Suppose V is a non-trivial proper linear subspace of $C^n$ such that

$$VM_o \subset V, \qquad VM_1 \subset V$$

There is at least one vector v in V such that the n-th component is not zero. For if $v=(v_1,\dots,v_{n-1},o)$ be a non-trivial vector in V, then from the condition $vM_o \in V$, we have;

$$(e_1v_1,e_2v_2,\dots,e_{n-1}v_{n-1},\ \sum_1^{n-1} q_j(e_j-1)v_j)$$

If V consists of the vectors whose n-th component is zero, then we have

$$(e_1v_1,\dots,e_{n-1}v_{n-1},\ o) \in V$$

$$\sum_{j=1}^{n-1} q_j(e_j-1)v_j = 0$$

Similarly, we have (n-1) conditions of the form:

$$\sum_{j=1}^{n-1} q_j(e_j-1)e_j^k v_j = 0 \qquad (k=1,\dots,n-1)$$

Since the Vandelmonde determinant $\det(e_j^k)$ is not zero under the condition $3^o$, we have identically:

$$q_j(e_j-1)v_j = 0 \qquad j:1,2,\dots,n-1.$$

That is , v is a trivial vector.

Suppose now $v=(v_1,\dots,v_{n-1},1)$ is in V. Then we have

$$(e_n-1)^{-1}v(M_1-I) = (p_1,\dots p_{n-1},1)\in V.$$

We claim that n row vectors $P=(p_1,\dots,p_{n-1},1),PM_o,PM_o^2,\dots,PM_o^{n-1}$ are linearly independent, that is, V is actually the full space. If we write the matrix $M_o$ in the form:

$$M_o = (I+C)^{-1}\mathrm{diag}.[e_1,\dots,e_{n-1},1](I+C)$$

with:

$$C = \begin{matrix} o & o & \cdots\cdots & q_1 \\ o & o & \cdots\cdots & q_2 \\ & & \cdots\cdots\cdots & \\ o & o & \cdots\cdots & o \end{matrix}$$

it is easy to see the k-th power $M_o^k$ to be of the form:

$$M_o^k = \begin{pmatrix} e_1^k & \cdots\cdots\cdots & & q_1(e_1^k - 1) \\ \cdot\cdot & e_2^k & \cdots\cdots & q_2(e_2^k - 1) \\ & \cdots\cdots\cdots & & \cdots \\ \cdots\cdots\cdots & & e_{n-1}^k & q_{n-1}(e_{n-1}^k - 1) \\ & \cdots\cdots\cdots\cdots & & 1 \end{pmatrix}$$

Now it is sufficient to show that the following determinant is not zero to prove the theorem:

$$\det \begin{pmatrix} P \\ PM_o \\ \cdot \\ \cdot \\ \cdot \\ PM_o^{n-1} \end{pmatrix} = \det. \begin{pmatrix} p_1 & p_2 & \cdots\cdots & 1 \\ p_1 e_1 & p_2 e_2 & & \Sigma\, p_j q_j (e_j - 1) + 1 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ p_1 e_1^{n-1} & p_2 e_2^{n-1} & \cdots\cdots & \Sigma\, p_j q_j (e_j^{n-1} - 1) + 1 \end{pmatrix}$$

$$= \det. \begin{pmatrix} p_1 & p_2 & \cdots\cdots & 1 - \Sigma\, p_j q_j \\ p_1 e_1 & p_2 e_2 & \cdots\cdots & 1 - \Sigma\, p_j q_j \\ & \cdots\cdots & \cdots\cdots & \\ p_1 e_1^{n-1} & p_2 e_2^{n-1} & \cdots\cdots & 1 - \Sigma\, p_j q_j \end{pmatrix}$$

$$= \det. \begin{pmatrix} e_1 - 1 & e_2 - 1 & \cdots\cdots & e_{n-1} - 1 \\ e_1^2 - 1 & e_2^2 - 1 & \cdots\cdots & e_{n-1}^2 - 1 \\ & \cdots\cdots & & \\ e_1^{n-1} - 1 & e_2^{n-1} - 1 & \cdots\cdots & e_{n-1}^{n-1} - 1 \end{pmatrix} \cdot p_1 p_2 \cdots\cdots p_{n-1} [\, 1 - \Sigma\, p_j q_j ]$$

$$= [\, 1 - \Sigma\, p_j q_j \,]\, p_1 p_2 \cdots p_{n-1} (e_1 - 1) \cdots\cdots (e_{n-1} - 1) V(e_1, \ldots, e_{n-1})$$

where $V(e_1, \ldots, e_{n-1})$ denotes ~~the~~ Vandelmonde determinant formed by n-1 quantities $e_1, \ldots, e_{n-1}$. If we use the formula :

$$1 - \sum_{j=1}^{n-1} p_j q_j = \prod_{j=1}^{n} [\sin \pi c_j] / [\sin \pi a_j]$$

with the conditions 1°, 2°, 3°, we see the determinant in question is not zero.

Theorem 5. Let B a diagonal matrix $B=\mathrm{diag}(\lambda_1,\dots,\lambda_n)$ with mutually distinct diagonal elements $\lambda_1,\dots,\lambda_n$. Let A be a matrix whose (j,k) elements is denoted by $a_{j,k}$ with eigenvalues $\rho_1,\rho_2,\dots,\rho_n$. We assume:

$1^\circ$. $a_{jj} \neq o$ (mod.1), $2^\circ$ $\exists$ $\rho_k \neq o$ (mod.1) , $3^\circ$ $\rho_k - a_{jj} \neq o$ (mod.1)

for all j,k. Then the system

$$(t-B)dx/dt = Ax$$

has a set of n solutions:

$$x_j(t) = (t-\lambda_j)^{a_{jj}} \sum_{m=o}^{\infty} g_j(m)(t-\lambda_j)^m \qquad (j=1,2,\dots,n)$$

which constitute a fundamental set of solutions X(t). The monodromy representation with respect to X(t) has the set of genrators:

$$M_j = I + (e_j-1)\begin{pmatrix} o & o & \cdots\cdots & o \\ & \cdots\cdots\cdots & \cdots & \\ p_{j1} & p_{j2} & \dots,1,\dots & p_{jn} \\ o & o & \cdots\cdots & o \end{pmatrix} \qquad (j=1,2,\dots,n)$$

Theorem 6. Besides the 3 conditions in Theorem 5, we assume: there is a number j such that $4^\circ$ $p_{j,k} \neq o$ for all k, $5^\circ$ $p_{k,j} \neq o$ for all k, $6^\circ$ the set of n vectors $P_k=(p_{k1},p_{k2},\dots\dots,p_{kn}),(k=1,2,\dots,n)$ are linearly independent. Then the system is irreducible.

Proof. It can be shown as in the proof of theorem 4, that there is at least one vector v whose j-th component is not zero in any invariant subspace V of $C_n$. Let this non-zero component be 1. Then we have $vM_j-v = (e_j-1)P_j$. That is to say $P_j$ is in V. Similarly, we have $P_jM_k=P_j+p_{j,k}(e_k-1)P_k$, and this shows $P_k$ is in V. Now the invariant subspace V conteins n linearly independent vectors. This shows that there is no non-trivial proper linear subspace V in variant under the monodromy. This completes the proof.